Cohomology of the Moduli Space of Stable Bundles

1 Overview

This article is a short review on the basics of Yang–Mills equations over a curve and the main ingredients of the calculations of Betti numbers of the moduli space of stable bundles.

2 Basic Setup

Suppose $P \xrightarrow{\pi} X$ a principal $G-$bundle.
Connections on a principal $G-$bundle can be seen in two ways:

1. Consider the vertical bundle $V = \ker\pi_* \subset TP$. Then a connection on P is just a choice of $G-$invariant horizontal bundle such that $TP = V \oplus H$.

2. Consider a $G-$equivariant $\mathfrak{g}-$valued one-form ω which satisfies $\omega(\sigma(X)) = X$ where $\sigma(X)$ is the fundamental vector field of X.

We can pass through the first to the second point of view by considering the one-form which is the projection to V. Conversely, we can consider $H = \ker\omega$.

Suppose ω, ω' connection forms, then $\omega - \omega'$ vanishes on the vertical bundle, and right-invariant, it is tensorial. So it descends to a $\mathfrak{g}_P-$valued one-form on X, where $\mathfrak{g}_P$ is $P \times_{\mathrm{ad}} \mathfrak{g}$, the adjoint bundle. So we see that the space of connections form an affine space, modelled on $\Omega^1(\mathfrak{g}_P)$.

We now consider gauge transformations, which are just automorphisms of the principal bundle. Suppose $\Phi \in \mathrm{Aut}P$. Then since $\Phi(p) = pf(p)$, where $f : P \to G$, so

$$\begin{aligned} pf(p)g &= \Phi(p)g \\ &= \Phi(pg) \\ &= pgf(pg). \end{aligned}$$

Since G acts freely, we have

$$f(pg) = g^{-1}f(p)g.$$

Hence f descends to a section of X to $\mathrm{Ad}P = P \times_{\mathrm{Ad}} G$. So $\mathrm{Aut}P \cong \Gamma\mathrm{Ad}P$.

The curvature of a connection is defined as $F_A = d\omega_A + \frac{1}{2}[\omega_A, \omega_A]$, where ω_A is the connection 1-form of A. We now consider a Lagrangian $\|F_A\|^2$, along with the action $S(A) = \int_X \|F_A\|^2$. Since the space of connection is affine, the tangent space is itself, and we apply variational techniques to determine the equations of motions, i.e. the Yang–Mills equations. Consider a family of connections $A_t = A + t\eta$, then $F_{A_t} = F_A + td_A\eta + \frac{1}{2}t^2[\eta, \eta]$. So

$$S(A_t) = S(A) + 2t \int_X \langle d_A\eta, F_A\rangle + t^2 \int_X (\|d_A\eta\|^2 + \langle F_A, [\eta, \eta]\rangle) + \cdots,$$

hence the Yang–Mills equations are $d_A^* F_A = 0$, which is just $d_A \star F_A = 0$. Along with the Bianchi identity $d_A F_A = 0$, one can see this is a generalisation of the Maxwell's equations.

3 Narasimhan–Seshadri Theorem

The motivation comes from the Kempf–Ness Theorem, which the simplest form states that $V^{st}/G_{\mathbb{C}} = \nu^{-1}(0)/G$ where V is a $G-$vector space, V^{st} is a choice of stable vectors, $G_{\mathbb{C}}$ the complexification of G and ν the moment map. The more general form relates the GIT quotient and the symplectic quotient. We expect this behaviour to still be somewhat true in this infinite dimensional scenario, by considering the space to be space of connection $\mathcal{A}$ and the group to be the gauge group $\mathcal{G}$. So first we have to introduce a symplectic structure on the space of connections.

Suppose X is a Riemann surface, then the symplectic form on $\mathcal{A}$ is given by

$$\omega(\alpha, \beta) = \int_X \alpha \wedge \beta,$$

where $\alpha, \beta \in \Omega^1(\mathfrak{g}_P)$.

Define $\nu : \mathcal{A} \to \Omega^1(\mathfrak{g}_P)$, by $A \mapsto F_A$. Then the differential of ν gives $(d\nu)_A(a) = d_A a$. The vector field associated to $\phi \in \Omega^0(\mathfrak{g}_P)$ is $-d_A\phi$. So since

$$\begin{aligned} i_{X_\phi}\omega(b) &= -\int_X d_A\phi \wedge b \\ &= \int_X \phi \wedge d_A b \\ &= \langle (d\nu)_A(b), \phi\rangle, \end{aligned}$$

we see that ν serves as a moment map.

Now suppose $G = U(n)$, and that E is an associated bundle of P, with ∇ a $U(n)-$connection, then E is a holomorphic vector bundle with $\nabla^{0,1} = \bar{\partial}$ by Newlander–Nirenberg. Conversely suppose $\bar{\partial}_{\mathcal{E}} = \bar{\partial} + \alpha$ in a unitary trivialization, by taking $\nabla^{1,0} = \partial - \alpha^\dagger$, we have $\nabla = \nabla^{1,0} + \bar{\partial}_{\mathcal{E}}$ a unitary connection compatible with the holomorphic structure. So we see that $\mathcal{A} \cong \mathcal{C}$, where $\mathcal{C}$ is the space of holomorphic structures. Since we considered $G = U(n)$, this isomorphism depends on the hermitian structure of the vector bundle. To solve this we consider the complexified gauge group $\mathcal{G}_{\mathbb{C}}$. As two hermitian structures are related by a complexified gauge action, the complexified gauge orbits are precisely the isomorphism classes of holomorphic structure. So we want to find $\mathcal{A}^{st} \subset \mathcal{A}$ such that $\mathcal{A}^{st}/\mathcal{G}_{\mathbb{C}} = \nu^{-1}(0)/\mathcal{G}$.

The Narasimhan–Seshadri theorem states that an irreducible holomorphic bundle $\mathcal{E}$ is stable if and only if there exists compatible unitary connection A with constant central curvature, i.e. $\star F_A = -2\pi i\mu(\mathcal{E})$ where $\mu(\mathcal{E}) = \frac{deg(\mathcal{E})}{rank(\mathcal{E})}$ denotes the slope of the holomorphic structure. A holomorphic bundle is defined to be stable if for all subbundles $\mathcal{F}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$. So we see that this result is similar to Kempf–Ness except that now we consider projectively flat connections.

4 Equivariant Cohomology

We want to study the group action on a space X, but sometimes the orbit space X/G is too horrible. We want to study well behaved group actions, so we try to construct a free action from the original action. Suppose X is a $G-$space. Then we consider G acting on the Borel space $X \times EG$ diagonally where EG is the universal $G-$bundle. So G acts freely on $X \times EG$, and we consider $X_G = X \times_G EG = (X \times EG)/G$. If G acts freely on X, then $X_G \to X/G$ has contractible fibre EG so $X_G \simeq X/G$. In the general case, the fibre over $x \in X$ is $EG/G_x = BG_x$ where G_x is the stabilizer at x. So we see that by replacing X/G by X_G, we are studying a better $G-$action which is free, and that when G acts freely on X, they coincide (homotopically). We define the equivariant cohomology of X to be $H^*_G(X) = H^*(X_G)$. As $H^*_G(\mathrm{pt}) = H^*(BG)$, so H^*_G is a functor from $G-$spaces to $H^*(BG)$ modules.

We want to relate the $\mathcal{G}-$equivariant cohomology and the $\mathcal{G}_{\mathbb{C}}-$equivariant cohomology. For $g \in GL(n, \mathbb{C}), g$ has a unique decomposition as pu where p is a positive definite hermitian matrix and u is unitary. So we can scale the eigenvalue of p all to 1 and so $GL(n, \mathbb{C})$ deformation retracts to $U(n)$. So we see that $\mathcal{G}_{\mathbb{C}}/\mathcal{G}$ is contractible. Assuming $\mathcal{G} \subset \mathcal{G}_{\mathbb{C}}$ is an admissible subgroup, then $\mathcal{G}_{\mathbb{C}} \to X \times_{\mathcal{G}} E\mathcal{G}_{\mathbb{C}} \to X \times_{\mathcal{G}_{\mathbb{C}}} E\mathcal{G}_{\mathbb{C}}$ forms a fibre bundle with contractible fibres, so $H^*_{\mathcal{G}}(X) = H^*_{\mathcal{G}_{\mathbb{C}}}(X)$.

To study the topology of the gauge group, we need to identify $B\mathcal{G}$. Take $\mathcal{G} \to \mathrm{Map}_G(P, EG) \to \mathrm{Map}_P(X, BG)$, which is a principal fibration which is locally trivial if BG paracompact and locally contractible. As $\mathrm{Map}_G(P, EG)$ contractible, we have $B\mathcal{G} = \mathrm{Map}_P(X, BG)$, where $\mathrm{Map}_P(X, BG)$ are the maps from X to BG which pull back EG to P. Thom's theorem gives

$$\mathrm{Map}(X, K(A, n)) = \prod_q K(H^q(X; A), n - q).$$

The Eilenberg–Maclane spaces satisfy $H^n(X; A) = [X, K(A, n)]$. Although $BU(n)$ not an Eilenberg–Maclane space, each Chern class c_i is in $H^{2i}(BU(n); \mathbb{Z}) = [BU(n), K(\mathbb{Z}, 2i)]$, so we can choose a map $c_i^\# : BU(n) \to K(\mathbb{Z}, 2i)$. We take

$$\prod_{i=1}^n c_i^\# : BU(n) \to \prod_{i=1}^n K(\mathbb{Z}, 2i),$$

which induces isomorphism on rational cohomology groups, so this map is a rational homotopy equivalence. So we can use Thom's theorem to calculate the Poincaré of $B\mathcal{G}$.

Now we consider everything over $\mathbb{Q}$. So $\mathrm{Map}(X, BU(n)) = \mathrm{Map}(X, \prod K(\mathbb{Z}, 2i))$.

Hence for X genus g, we have

$$\begin{aligned}\mathrm{Map}(X, BU(n)) &= \prod_{i=1}^{n} \mathrm{Map}(X, K(\mathbb{Z}, 2i)) \\ &= \prod_{i=1}^{n} K(\mathbb{Z}, 2i) \times K(\mathbb{Z}, 2i-1)^{2g} \times K(\mathbb{Z}, 2i-2) \\ &= \mathbb{Z} \times K(\mathbb{Z}, 2n) \times \prod_{i=1}^{n-1} K(\mathbb{Z}, 2i)^2 \times \prod_{i=1}^{n} K(\mathbb{Z}, 2i-1)^{2g}.\end{aligned}$$

We have $K(\mathbb{Z}, 1) = S^1$ and $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$, so by the Leray–Serre spectral sequence, we have the Poincaré polynomials $P_q(K(\mathbb{Z}, 2k)) = \frac{1}{1-q^{2k}}$, $P_q(K(\mathbb{Z}, 2k-1)) = 1+q^{2k-1}$. Hence we see that $K(\mathbb{Z}, n)$ path connected for $n \neq 0$. So since for a fixed P, $\mathrm{Map}_P(X, BU(n)$ is just one path connected component of $\mathrm{Map}(X, BU(n))$, $\mathrm{Map}_P(X, BU(n))$ is just picking one entry in $\mathbb{Z}$, which corresponds to the first Chern class. Therefore, we have

$$P_q(B\mathcal{G}) = P_q(\mathrm{Map}_P(X, BU(n))) = \frac{\prod_{k=1}^{n}(1+q^{2k-1})^{2g}}{(1-q^{2n})\prod_{k=1}^{n-1}(1-q^{2k})^2}.$$

We will later use this to calculate the Poincaré polynomials of the moduli space of stable bundles with fixed rank and degree.

5 Topology of the Moduli Space of Stable Bundles

To calculate the betti numbers of the moduli space, we are going to introduce a stratification $\mathcal{A}_\mu$ on the space of connections. The stratification is indexed by a partially ordered set I. However, for the stratification to be useful in calculations, we need three properties:

1. $\overline{\mathcal{A}}_\nu \subset \cup_{\mu \geq \nu} \mathcal{A}_\mu$,
2. for finite $J \subset I$, there is a finite number of minimal elements in $I \setminus J$,
3. for each $q \in \mathbb{N}$ there are finite $\mu \in I$ such that $\mathrm{codim}(\mathcal{A}_\mu) < q$.

We define the stratification by the Harder–Narasimhan filtration. Any holomorphic bundle $\mathcal{E}$ admits a canonical Harder–Narasimhan filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \mathcal{F}_r = \mathcal{E},$$

with$\mathcal{D}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ semistable and that $\mu(\mathcal{D}_1) > \cdots > \mu(\mathcal{D}_r)$. So we define the Harder–Narasimhan vector $\mu = (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2, \ldots, \mu_r, \ldots, \mu_r)$ where $\mu_i = \mu(\mathcal{D}_i)$ and that μ_i appears $\mathrm{rank}\mathcal{D}_i$ times. Since the slopes are decreasing, if we plot out the graph of straight lines with slope μ_i's for $\mathrm{rank}\mathcal{D}_i$ units, we see that the graph is convex. We define that $\mu \geq \nu$ if the graph of μ lies completely above the graph of ν. Then we see that this stratification satisfies the above three properties.

To get the codimension of $\mathcal{A}_\mu$ we use Riemann Roch:

$$h^0(V) - h^1(V) = 2c_1(V) + \mathrm{rank}(V)(2-2g),$$

where h's denote the real dimension. We take $\mathrm{End}_0\mathcal{E}$ to be the bundle of endomorphisms which preserve filtration, and that $\mathrm{End}_1\mathcal{E} = \mathrm{End}\mathcal{E}/\mathrm{End}_0\mathcal{E}$. We have $H^0(\mathrm{End}_1\mathcal{E}) = 0$. As smooth bundles we also see that

$$\mathrm{End}_1\mathcal{E} \cong \bigoplus_{j>i} \mathcal{D}_j^* \otimes \mathcal{D}_i.$$

Hence we see that

$$h^1(\mathrm{End}_1\mathcal{E}) = 2\sum_{i>j}((n_ik_j - n_jk_i) + n_in_j(g-1)),$$

where n_i is the rank of $\mathcal{D}_i$ and k_i is the degree. Recall that if we have a first order deformation of transition functions α^i_j, which is a cocycle $\phi^i_j = \alpha^i_j + \beta^i_j\epsilon$. So the cocycle condition gives $\beta^k_j = \beta^i_j\alpha^k_i + \alpha^i_j\beta^k_i$. So if it is a deformation of the trivial bundle, we have the cocycle condition $\beta^k_j = \beta^i_j + \beta^k_i$, which gives a 1-cocycle with coefficients in the endomorphism bundle. So the 1-cocycles are in bijection with the first order deformation of a framed vector bundle, and since it turns out that two deformations are isomorphic iff the cocycles are cohomologous, H^1 of the endomorphism bundle classifies the deformations of the bundle. Hence h^1 is also the codimension of $\mathcal{A}_\mu$ since $H^1(\mathrm{End}\mathcal{E})$ is the deformation with $H^1(\mathrm{End}_0\mathcal{E})$ along the tangent directions to $\mathcal{A}_\mu$ and $H^1(\mathrm{End}_1\mathcal{E})$ along the directions complementary to $\mathcal{A}_\mu$.

We now fix a slope vector and take F_μ to be a bundle with fibre $GL(n,\mathbb{C})/B_\mu$ where B_μ is the subgroup such that it preserves a flag which matches the dimensions of the slope vector. So a section of F_μ is a filtration and take $\mathcal{F}_\mu$ to be the space of sections which matches the degree of the filtration. Since the Harder–Narasimhan filtration is canonical, we have a map $F : \mathcal{A}_\mu \to \mathcal{F}_\mu$. F is continuous and extends to Sobolev completions. Fix $E_\mu \in \mathcal{F}_\mu$, take $\mathcal{B}_\mu = F^{-1}(E_\mu)$, $\mathrm{Aut}(E_\mu)$ subgroup of $\mathcal{G}_\mathbb{C}$ preserving the filtration, so $\mathcal{F}_\mu = \mathcal{G}_\mathbb{C}/\mathrm{Aut}(E_\mu)$. We also see that $\mathcal{A}_\mu = \mathcal{G}_\mathbb{C} \times_{\mathrm{Aut}(E_\mu)} B_\mu$, so

$$\mathcal{A}_\mu \times_{\mathcal{G}_\mathbb{C}} E\mathcal{G}_\mathbb{C} = \mathcal{B}_\mu \times_{\mathrm{Aut}E_\mu} E\mathcal{G}_\mathbb{C} = \mathcal{B}_\mu \times_{\mathrm{Aut}E_\mu} E\mathrm{Aut}E_\mu.$$

Now we use Hermitian metric on E to take $\mathcal{F}_i = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_i$ and $\mathcal{B}^0_\mu \subset \mathcal{B}_\mu$ such that $\mathcal{D}_i$ semistable holomorphic and $\mathrm{Aut}(E^0_\mu)$ automorphisms preserving the splitting. So we have

$$\mathrm{Aut}(E^0_\mu) \cong \prod \mathrm{Aut}(\mathcal{D}_i)$$
$$\mathcal{B}^0_\mu \cong \prod \mathcal{A}^{ss}(\mathcal{D}_i),$$

where $\mathcal{A}^{ss}$ is the space of semistable connections. We see that $\mathrm{Aut}(E_\mu)$ deformation retracts to $\mathrm{Aut}(E_\mu)$ and $\mathcal{B}_\mu$ deformation retracts to $\mathcal{B}^0_\mu$ by scaling the off diagonal entries and the second fundamental form respectively. So we see that the Borel spaces

$$(\mathcal{B}^0_\mu)_{\mathrm{Aut}E^0_\mu} = (\mathcal{B}_\mu)_{\mathrm{Aut}E_\mu} = (\mathcal{A}_\mu)_{\mathcal{G}_\mathbb{C}}.$$

So over $\mathbb{Q}$, we have

$$H^*_{\mathcal{G}_\mathbb{C}}(\mathcal{A}_\mu) \cong \bigotimes H^*_{\mathrm{Aut}(\mathcal{D}_i)}(\mathcal{A}^{ss}(\mathcal{D}_i)),$$

by the Künneth formula. The stratification is $\mathcal{G}-$equivariantly perfect, so

$$H^*(B\mathcal{G}) = H^*(B\mathcal{G}_{\mathbb{C}}) = H^*_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}) = \sum_{\mu} H^{*-d_\mu}_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}_\mu),$$

with d_μ the codimension. So we obtained the relation

$$H^*(B\mathcal{G}) = \sum_{\mu} \bigotimes H^{*-d_\mu}_{\mathrm{Aut}(\mathcal{D}_i)}(\mathcal{A}^{ss}(\mathcal{D}_i)).$$

Suppose $gA = A$ for A a compatible connection on a stable bundle, this gives a holomorphic automorphisms so if not a constant scalar, this decomposes the bundle into proper holomorphic eigensubbundles $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$. As $\mathcal{E}$ stable, this is a contradiction since both $\mathcal{E}_1$ and $\mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_k$ has both strictly smaller and larger slope than $\mathcal{E}$. So we see that the stabilizer of a compatible connection on a stable bundle under $\mathcal{G}_{\mathbb{C}}$ action consists of constant central scalars. So we see that $\mathcal{G}_{\mathbb{C}}/\mathbb{C}^*-$equivariant cohomology of the space of stable connections $\mathcal{A}^{st}$ is the actual cohomology of $\mathcal{A}^{st}/\mathcal{G}_{\mathbb{C}}$. We shall see now how to recover the actual cohomology from the $\mathcal{G}_{\mathbb{C}}-$equivariant cohomology. Consider $\mathbb{C}^* \to \mathcal{G}_{\mathbb{C}} \to \mathcal{G}_{\mathbb{C}}/\mathbb{C}^*$. We fix a frame, and take the determinant of a gauge transformation to get a map $\mathcal{G}_{\mathbb{C}} \to \mathbb{C}^*$. The map $\mathbb{C}^* \to \mathcal{G}_{\mathbb{C}} \to \mathbb{C}^*$ is precisely the map $z \mapsto z^n$, so it is "rationally a splitting", i.e. over $\mathbb{Q}$, we have $\mathcal{G}_{\mathbb{C}} = \mathcal{G}_{\mathbb{C}}/\mathbb{C}^* \times \mathbb{C}^*$. So we see that

$$H^*_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}^{st}) = H^*(\mathcal{A}^{st}/\mathcal{G}_{\mathbb{C}}) \otimes H^*(B\mathbb{C}^*).$$

The technical foundations are more or less in place now. We are ready to see how these results are used in the computation of Poincaré polynomials of the moduli spaces.

6 Examples

Now we compute some examples using the results we outlined above. The computation of the Poincaré polynomials for the moduli space of stable bundles $\mathcal{M}^{st}(n,k)$ of rank n, degree k where $(n,k) = 1$ is an inductive procedure. We will illustrate this by computing the Poincaré polynomials of $\mathcal{M}^{st}(2,1)$ and $\mathcal{M}^{st}(3,1)$.

6.1 $\mathcal{M}^{st}(2,1)$

Suppose $n = 2$, $k = 1$. Then we have

$$P_q(B\mathcal{G}) = \frac{(1+q)^{2g}(1+q^3)^{2g}}{(1-q^2)^2(1-q^4)},$$

and that we have the possible slope vectors are $(\frac{1}{2}, \frac{1}{2})$ and $(l+1, -l)$ where $l \geq 0$. The first case corresponds to the stable case, and the second case corresponds to the Harder–Narasimhan filtration $0 \subset \mathcal{F} \subset \mathcal{E}$. Since any holomorphic line bundle is stable, we have

$$\begin{aligned} H^*_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}_l) &= H^*_{\mathrm{Aut}(\mathcal{F})}(\mathcal{A}^{st}(\mathcal{F})) \otimes H^*_{\mathrm{Aut}(\mathcal{E}/\mathcal{F})}(\mathcal{A}^{st}(\mathcal{E}/\mathcal{F})) \\ &= H^*(B\mathrm{Aut}(\mathcal{F})) \otimes H^*(B\mathrm{Aut}(\mathcal{E}/\mathcal{F})). \end{aligned}$$

So we see that

$$P_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}_l) = \left(\frac{(1+q)^{2g}}{1-q^2}\right)^2$$

and the formula for the codimension gives

$$d_\mu = 4l + 2g,$$

so we have

$$\begin{aligned} P_q(B\mathcal{G}) &= \frac{(1+q)^{2g}(1+q^3)^{2g}}{(1-q^2)^2(1-q^4)} \\ &= P_{\mathcal{G}_\mathbb{C}}(\mathcal{A}^{st}) + \sum_{l=0}^{\infty} q^{4l+2g} \left(\frac{(1+q)^{2g}}{1-q^2} \right)^2 \\ &= P_{\mathcal{G}_\mathbb{C}}(\mathcal{A}^{st}) + \left(\frac{q^{2g}}{1-q^4} \right) \left(\frac{(1+q)^{2g}}{1-q^2} \right)^2, \end{aligned}$$

Hence we obtain

$$P_{\mathcal{G}_\mathbb{C}}(\mathcal{A}^{st}) = (1+q)^{2g} \frac{(1+q^3)^{2g} - q^{2g}(1+q)^{2g}}{(1-q^2)^2(1-q^4)}.$$

Since we know that $H^*_{\mathcal{G}_\mathbb{C}}(\mathcal{A}^{st}) = H^*(\mathcal{A}^{st}/\mathcal{G}_\mathbb{C}) \otimes H^*(B\mathbb{C}^*)$, there is an additional $\frac{1}{1-q^2}$ factor, hence

$$P_q(\mathcal{M}^{st}(2,1)) = (1+q)^{2g} \frac{(1+q^3)^{2g} - q^{2g}(1+q)^{2g}}{(1-q^2)(1-q^4)}.$$

6.2 $\mathcal{M}^{st}(3,1)$

Suppose $n=3$, $k=1$. Then we have

$$P_q(B\mathcal{G}) = \frac{(1+q)^{2g}(1+q^3)^{2g}(1+q^5)^{2g}}{(1-q^2)^2(1-q^4)^2(1-q^6)},$$

and that we have the possible slope vectors $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(s+1, -\frac{s}{2}, -\frac{s}{2})$, $(\frac{r+1}{2}, \frac{r+1}{2}, -r)$ and $(r+s+1, -r, -s)$. So we see that to do the calculation we have to calculate the Poincaré polynomial for $\mathcal{A}^{ss}$ when $n=2$ and k arbitrary, the only thing we lost when k is even is that semistable does not imply stable but the calculation is the same, since we only use strictly stable to calculate the actually homology of the moduli space but since we work in the equivariant setting everything proceeds the same. So we have

$$\begin{aligned} P_{\mathcal{G}_\mathbb{C}}(\mathcal{A}^{ss}) &= \frac{(1+q)^{2g}(1+q^3)^{2g}}{(1-q^2)^2(1-q^4)} - \left(\frac{q^{2g+2k+\lfloor -\frac{k}{2} \rfloor -1}}{1-q^4} \right) \left(\frac{(1+q)^{2g}}{1-q^2} \right)^2 \\ &= (1+q)^{2g} \frac{(1+q^3)^{2g} - q^{2g+2k+\lfloor -\frac{k}{2} \rfloor -1}(1+q)^{2g}}{(1-q^2)^2(1-q^4)}. \end{aligned}$$

So as before we have

$$
\begin{aligned}
P_q(B\mathcal{G}) &= \frac{(1+q)^{2g}(1+q^3)^{2g}(1+q^5)^{2g}}{(1-q^2)^2(1-q^4)^2(1-q^6)} \\
&= P_{\mathcal{G}_{\mathbb{C}}}(\mathcal{A}^{st}) + \sum_{s=0}^{\infty} q^{6s+4g}\left(\frac{(1+q)^{2g}}{1-q^2}\right)(1+q)^{2g}\frac{(1+q^3)^{2g} - q^{2g-2s+\lfloor\frac{s}{2}\rfloor-1}(1+q)^{2g}}{(1-q^2)^2(1-q^4)} \\
&+ \sum_{r=0}^{\infty} q^{6r+4g-2}\left(\frac{(1+q)^{2g}}{1-q^2}\right)(1+q)^{2g}\frac{(1+q^3)^{2g} - q^{2g+2r+\lfloor-\frac{r+1}{2}\rfloor+1}(1+q)^{2g}}{(1-q^2)^2(1-q^4)} \\
&+ \sum_{s=0}^{\infty}\sum_{r=\lfloor-\frac{s+1}{2}\rfloor+1}^{s} q^{8r+8s+6g-2}\left(\frac{(1+q)^{2g}}{1-q^2}\right)^3,
\end{aligned}
$$

and from this we can calculate the Poincaré polynomial of $\mathcal{M}^{st}(3,1)$ by multiplying a factor of $1-q^2$.

References

[AB] Michael Atiyah, Raoul Bott, The Yang-Mills equations over Riemann surfaces, *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 308, No. 1505 (Mar. 17, 1983), pp. 523-615.

[D] Simon Donaldson. A new proof of a theorem of Narasimhan and Seshadri, *J. Differential Geom.*, Volume 18, Number 2 (1983), 269-277.

[E] Jonny Evans. Aspects of Yang–Mills Theory. *lecture notes*.